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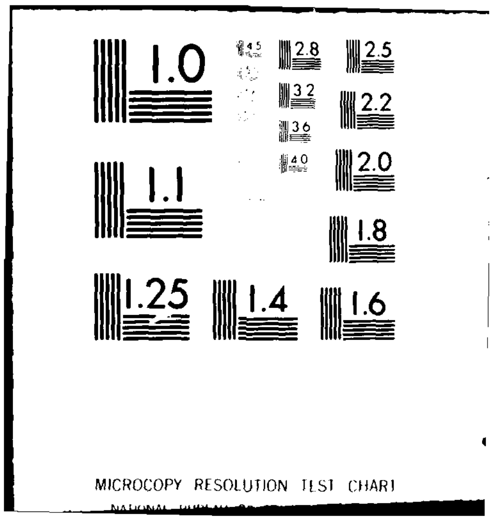
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RELATIONSHIPS BETWEEN ELEMENTS OF THE STOKES MATRIX

AD A094188

BY

Edward S. Fry and George W. Kattawar

Report No. 15

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THE STOKES MATRIX

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ABSTRACT

Although there are sixteen elements of the Stokes Matrix, they are constructed from basically four amplitudes and three phase differences. This of course implies that there exists nine independent relationships connecting the elements. These relationships are equalities for scattering by a single particle in a fixed orientation and in a fixed direction. They become inequalities when Stokes matrices from an ensemble of particles differing in size, orientation, morphology, or optical properties are added incoherently. These relations will prove to be very useful for providing consistency checks on experimental measurements of all sixteen elements.

I. Introduction

In the classic work of Van de Hulst[1], he points out the fact that there exists nine independent relationships (equalities) between the sixteen elements of the 4x4 Stokes matrix. These relations however were not presented. He also stated that when Stokes matrices were added incoherently that the equalities became inequalities. This situation arises when one is measuring the combined Stokes matrix from an ensemble of particles differing in size, orientation, morphology or optical properties.

Experimentalists are now routinely measuring <u>all</u> sixteen elements of the Stokes matrix for both single particles and collections of particles.[2-4] It is therefore important that these relations be employed routinely to check for consistency. To our knowledge the only authors who have presented the equalities explicitly were Abhyankar and Fymat.[5] We will show that the first six of their relations (10c - 15c), which are quadratic in the elements are correct, however their remaining three equations (19c - 21c) are quartic relations which we will show are actually the product of <u>two</u> quadratic equalities, one of which is <u>not</u> independent and is therefore redundant. We will also present proofs for the inequalities along with examples.

II. DERIVATION OF EQUALITIES

We will use the notation of Van de Hulst[1] in the following derivations. Let the complex scattered amplitudes be written as

$$A_j = \alpha_j e^{i\beta_j}, \; j = 1,2,3,4. \qquad (1)$$

and let

$$\epsilon = \beta_1 - \beta_2, \tag{2a}$$

$$\delta = \beta_3 - \beta_2, \tag{2b}$$

$$\gamma = \beta_1 - \beta_4, \tag{2c}$$

$$\sigma = \beta_4 - \beta_2, \tag{2d}$$

$$\lambda = \beta_1 - \beta_3, \tag{2e}$$

$$\eta = \beta_4 - \beta_3. \tag{2f}$$

With this notation the elements of the Stokes matrix (f_{ij}) can be written as follows:

$$f_{11} = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2)/2, \tag{3a}$$

$$f_{12} = (-\alpha_1^2 + \alpha_2^2 - \alpha_3^2 + \alpha_4^2)/2, \tag{3b}$$

$$f_{13} = \alpha_2\alpha_3 \cos\delta + \alpha_1\alpha_4 \cos\gamma, \tag{3c}$$

$$f_{14} = - \alpha_2\alpha_3 \sin\delta - \alpha_1\alpha_4 \sin\gamma, \tag{3d}$$

$$f_{21} = (-\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \alpha_4^2)/2, \tag{3e}$$

$$f_{22} = (\alpha_1^2 + \alpha_2^2 - \alpha_3^2 - \alpha_4^2)/2, \tag{3f}$$

$$f_{23} = \alpha_2\alpha_3 \cos\delta - \alpha_1\alpha_4 \cos\gamma, \tag{3g}$$

$$f_{24} = -\alpha_2\alpha_3 \sin\delta + \alpha_1\alpha_4 \sin\gamma, \tag{3h}$$

$$f_{31} = \alpha_2\alpha_4 \cos\sigma + \alpha_1\alpha_3 \cos\lambda, \tag{3i}$$

$$f_{32} = \alpha_2\alpha_4 \cos\sigma - \alpha_1\alpha_3 \cos\lambda, \tag{3j}$$

$$f_{33} = \alpha_1\alpha_2 \cos\epsilon + \alpha_3\alpha_4 \cos\eta, \tag{3k}$$

$$f_{34} = -\alpha_1\alpha_2 \sin\epsilon + \alpha_3\alpha_4 \sin\eta, \tag{3l}$$

$$f_{41} = \alpha_2\alpha_4 \sin\sigma + \alpha_1\alpha_3 \sin\lambda, \tag{3m}$$

$$f_{42} = \alpha_2\alpha_4 \sin\sigma - \alpha_1\alpha_3 \sin\lambda, \tag{3n}$$

$$f_{43} = \alpha_1\alpha_2 \sin\epsilon + \alpha_3\alpha_4 \sin\eta, \tag{3o}$$

$$f_{44} = \alpha_1\alpha_2 \cos\epsilon - \alpha_3\alpha_4 \cos\eta. \tag{3p}$$

With the elements written in the above form, it is quite straight forward to obtain the following nine independent equalities:

$$(f_{11} + f_{22})^2 - (f_{12} + f_{21})^2 = (f_{33} + f_{44})^2 + (f_{43} - f_{34})^2 = 4\alpha_1{}^2\alpha_2{}^2, \quad (4a)$$

$$(f_{11} - f_{22})^2 - (f_{21} - f_{12})^2 = (f_{33} - f_{44})^2 + (f_{43} + f_{34})^2 = 4\alpha_3{}^2\alpha_4{}^2, \quad (4b)$$

$$(f_{11} + f_{21})^2 - (f_{12} + f_{22})^2 = (f_{13} + f_{23})^2 + (f_{14} + f_{24})^2 = 4\alpha_2{}^2\alpha_3{}^2 \quad (4c)$$

$$(f_{11} - f_{21})^2 - (f_{12} - f_{22})^2 = (f_{13} - f_{23})^2 + (f_{14} - f_{24})^2 = 4\alpha_1{}^2\alpha_4{}^2, \quad (4d)$$

$$(f_{11} + f_{12})^2 - (f_{21} + f_{22})^2 = (f_{31} + f_{32})^2 + (f_{41} + f_{42})^2 = 4\alpha_2{}^2\alpha_4{}^2, \quad (4e)$$

$$(f_{11} - f_{12})^2 - (f_{21} - f_{22})^2 = (f_{31} - f_{32})^2 + (f_{41} - f_{42})^2 = 4\alpha_1{}^2\alpha_3{}^2, \quad (4f)$$

$$f_{13}\, f_{14} - f_{23}\, f_{24} = f_{33}\, f_{34} + f_{43}\, f_{44} = -2\alpha_1\alpha_2\alpha_3\alpha_4 \sin(\beta_1-\beta_2+\beta_3-\beta_4), \quad (4g)$$

$$f_{14}\, f_{23} - f_{13}\, f_{24} = f_{42}\, f_{31} - f_{41}\, f_{32} = -2\alpha_1\alpha_2\alpha_3\alpha_4 \sin(\beta_1+\beta_2-\beta_3-\beta_4), \quad (4h)$$

$$f_{31}\, f_{41} - f_{32}\, f_{42} = f_{33}\, f_{43} + f_{34}\, f_{44} = 2\alpha_1\alpha_2\alpha_3\alpha_4 \sin(\beta_1-\beta_2-\beta_3+\beta_4). \quad (4i)$$

In place of eqns. (4g) - (4i) we can use the following three independent equations, namely,

$$f_{33}{}^2 - f_{34}{}^2 + f_{43}{}^2 - f_{44}{}^2 = f_{13}{}^2 - f_{14}{}^2 - f_{23}{}^2 + f_{24}{}^2 = 4\alpha_1\alpha_2\alpha_3\alpha_4 \cos(\beta_1-\beta_2+\beta_3-\beta_4), \quad (5a)$$

$$f_{33}{}^2 - f_{43}{}^2 + f_{34}{}^2 - f_{44}{}^2 = f_{31}{}^2 - f_{41}{}^2 - f_{32}{}^2 + f_{42}{}^2 = 4\alpha_1\alpha_2\alpha_3\alpha_4 \cos(\beta_1-\beta_2-\beta_3+\beta_4), \quad (5b)$$

$$f_{31}{}^2 - f_{32}{}^2 + f_{41}{}^2 - f_{42}{}^2 = f_{14}{}^2 - f_{24}{}^2 + f_{13}{}^2 - f_{23}{}^2 = 4\alpha_1\alpha_2\alpha_3\alpha_4 \cos(\beta_1+\beta_2-\beta_3-\beta_4). \quad (5c)$$

It should be noted that eqns. (4a) - (4f) were also obtained by Abhyankar and Fymat[5]; however, eqns. (4g) - (4i) and (5a) - (5c) were given as products. For example their eqn. (19c) was given as the product of our eqns. (4g) and (5a) yielding a quartic relation, which we now see is unnecessary. In fact, it is not too difficult to show that the square of the lhs of eqn. (5a) is equal to the product of the middle termsof eqns. (4c) and (4d) minus four times the square of the middle term of eqn. (4g).

An interesting equality can be obtained by summing eqns. (4a) - (4f) namely

$$\sum_{i,j} f_{ij}^2 = 4f_{11}^2, \; i,j = 1,2,3,4. \tag{6}$$

It is interesting to note that for a spherical particle where

$$f_{31} = f_{32} = f_{13} = f_{23} = f_{14} = f_{41} = f_{42} = f_{24} = 0 \; , \tag{7a}$$

and

$$f_{11} = f_{22}, \; f_{12} = f_{21}, \; f_{33} = f_{44}, \; f_{34} = -f_{43}, \tag{7b}$$

that only one non-trivial equality exists which is eqn. (4a) which yields

$$f_{11}^2 = f_{12}^2 + f_{33}^2 + f_{34}^2 \tag{8}$$

III. DERIVATION OF INEQUALITIES

We now would like to consider what happens when we add incoherently the Stokes matrices from an ensemble of particles differing in size, orientation, morphology or optical properties. We will use a superscript to denote scattering amplitudes and phases for each member of the ensemble. With this notation the lhs (left hand side) of eqn. (4a) becomes

$$(f_{11}^T + f_{22}^T)^2 - (f_{12}^T + f_{21}^T)^2 = 4\sum_{i,j} \alpha_1^i \alpha_1^i \alpha_2^j \alpha_2^j \; , \tag{9a}$$

whereas the rhs becomes

$$(f_{33}^T + f_{44}^T)^2 + (f_{43}^T - f_{34}^T)^2 = 4\sum_{i,j} \alpha_1^i \alpha_2^i \alpha_1^j \alpha_2^j \cos(\epsilon^i - \epsilon^j). \tag{9b}$$

It should be understood that f_{kl}^T as used here denoteselements of the total Stokes matrix of the ensemble and hence the use of the superscript T and also that the summation indices i and j are over all members comprising the ensemble. Now the rhs of eqn. (9a) can be written as

$$4\sum_{i,j} \alpha_1^i \alpha_1^i \alpha_2^j \alpha_2^j = 2\sum_{i,j} (\alpha_1^i \alpha_1^i \alpha_2^j \alpha_2^j + \alpha_1^j \alpha_1^j \alpha_2^i \alpha_2^i)$$

$$= 2\sum_{i,j} (\alpha_1^i \alpha_2^j - \alpha_1^j \alpha_2^i)^2 + 4\sum_{i,j} \alpha_1^i \alpha_2^i \alpha_1^j \alpha_2^j$$

$$\geq 4\sum_{i,j} \alpha_1^i \alpha_2^i \alpha_1^j \alpha_2^j \geq 4\sum_{i,j} \alpha_1^i \alpha_2^i \alpha_1^j \alpha_2^j \cos(\epsilon^i - \epsilon^j) \tag{9c}$$

which is the rhs of eqn. (9b); and, therefore

$$(f_{11}^T + f_{22}^T)^2 - (f_{12}^T + f_{21}^T)^2 \geq (f_{33}^T + f_{44}^T)^2 + (f_{43}^T - f_{34}^T)^2. \tag{9d}$$

It is now clear that the inequalities corresponding to eqns. (4a)-(4f) become

$$(f_{11}^T + f_{22}^T)^2 - (f_{12}^T + f_{21}^T)^2 \geq (f_{33}^T + f_{44}^T)^2 + (f_{43}^T - f_{34}^T)^2, \tag{10a}$$
$$(f_{11}^T - f_{22}^T)^2 - (f_{21}^T - f_{12}^T)^2 \geq (f_{33}^T - f_{44}^T)^2 + (f_{43}^T + f_{34}^T)^2, \tag{10b}$$
$$(f_{11}^T + f_{21}^T)^2 - (f_{12}^T + f_{22}^T)^2 \geq (f_{13}^T + f_{23}^T)^2 + (f_{14}^T + f_{24}^T)^2, \tag{10c}$$
$$(f_{11}^T - f_{21}^T)^2 - (f_{12}^T - f_{22}^T)^2 \geq (f_{13}^T - f_{23}^T)^2 + (f_{14}^T - f_{24}^T)^2, \tag{10d}$$
$$(f_{11}^T + f_{12}^T)^2 - (f_{21}^T + f_{22}^T)^2 \geq (f_{31}^T + f_{32}^T)^2 + (f_{41}^T + f_{42}^T)^2, \tag{10e}$$
$$(f_{11}^T - f_{12}^T)^2 - (f_{21}^T - f_{22}^T)^2 \geq (f_{31}^T - f_{32}^T)^2 + (f_{41}^T - f_{42}^T)^2. \tag{10f}$$

Now the inequality for eqn. (6) follows immediately and is

$$\sum_{i,j} (f_{ij}^T)^2 \leq 4(f_{11}^T)^2 \quad i,j = 1,2,3,4. \tag{11}$$

One-way inequalities for eqns. (4g) - (4i) and (5a) - (5c) cannot be established. For example the lhs of eqn. (4g) yields

$$2 \sum_{ij} \alpha_1{}^j \alpha_2{}^i \alpha_3{}^j \alpha_4{}^i \sin (\beta_1{}^j - \beta_2{}^i + \beta_3{}^j - \beta_4{}^i), \tag{12}$$

whereas the middle term of eqn. (4g) yields

$$2 \sum_{ij} \alpha_1{}^i \alpha_2{}^i \alpha_3{}^j \alpha_4{}^j \sin (\beta_1{}^j - \beta_2{}^j + \beta_3{}^i - \beta_4{}^i), \tag{13}$$

and we can see that the inequality can go in either direction depending on the amplitudes and relative phase difference at each scattering angle of particles in the ensemble.

It should also be noted that the inequalities derived above also hold if one makes measurements on a single particle but averages over a finite solid angle.

IV. Test Of Inequalities

To validate the inequalities we have presented, we performed the following two calculations. We first computed the elements of the Stokes matrix for a single sphere of size parameter 7.9109 with a real refractive index of 1.6146. The results were then averaged over a 3^o angular range. We then tested eqn. (10) by using the normalized Stokes matrix i.e., we define $F_{ij} = f_{ij}/f_{11}$ and then eqn. (10) becomes

$$\sum_{i,j}(F_{ij}^T)^2 \leq 4 \; ; \; i,j = 1,2,3,4. \qquad (14)$$

The results of this calculation are shown in Fig. 1 along with the only non-zero normalized elements (with the obvious exception of $F_{11} = 1$ for all scattering angles) $F_{12} = F_{21}$, $F_{33} = F_{44}$, and $F_{43} = -F_{34}$. It is immediately clear that there is a great deal of structure in the deviation of $\sum_{i,j} F_{ij}^2$ from four. In fact a deviation as large as 18% occurs at a scattering angle of ~ 157^o.

The next case we considered was to average over a polydispersion of spheres and test the inequality with no angular averaging. To do this we assumed a Gaussian distribution of spherical particles with a mean size parameter of 7.9109, the same as for the single sphere in Fig. 1, and a standard derivation of 0.117 in size parameter and the same refractive index as used in the single sphere case. These results are shown in Fig. 2. What is particularly noteworthy is the fact that even though the normalized elements are quite similar between the two cases, the $\sum_{i,j} F_{ij}^2$ is quite dissimilar in structure.

Due to the quadratic nature of the inequalities, we conjecture that the use of them may provide a more discriminating test of small differences between Stokes matrix elements of two slightly dissimilar objects. Experimental verification of this idea is presently being pursued and the results will appear in a future publication.

V. Conclusion

We have presented both the equalities and inequalities which occur between the sixteen elements of the Stokes matrix. The use of them should become a routine part of any experiment designed to measure them. The reason being that they can be used as a self consistency test on the experimental measurements. For example if one finds that any one of the inequalities is violated, then one can safely assume that there are noise and/or systematic problems in the measurements. This appears far superior than simply trying to fit some theoretical curve to the data in order to determine the noise level. The inequalities also hold for measurements made on a single member of an ensemble averaged over a finite solid angle which is always realized experimentally. We would also like to make a conjecture that the inequality tests may provide a very useful way for discriminating between two slightly dissimilar objects.

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FIGURE CAPTIONS

Fig. 1 Plot of $\sum_{i,j} F_{ij}^2$; i,j = 1,2,3,4, for a single sphere as a function of scattering angle for a 3^o angular average. The computations were carried out using Mie theory for a size parameter of 7.9109. and a real refractive index of 1.6146. Also shown are the normalized elements F_{12}, F_{33}, and F_{43}.

Fig. 2 Same as Fig. 1 except a Gaussian size distribution of spheres was used with a mean size parameter of 7.9109and a standard deviation of the size parameter of 0.117. No angular averaging was done for this calculation.

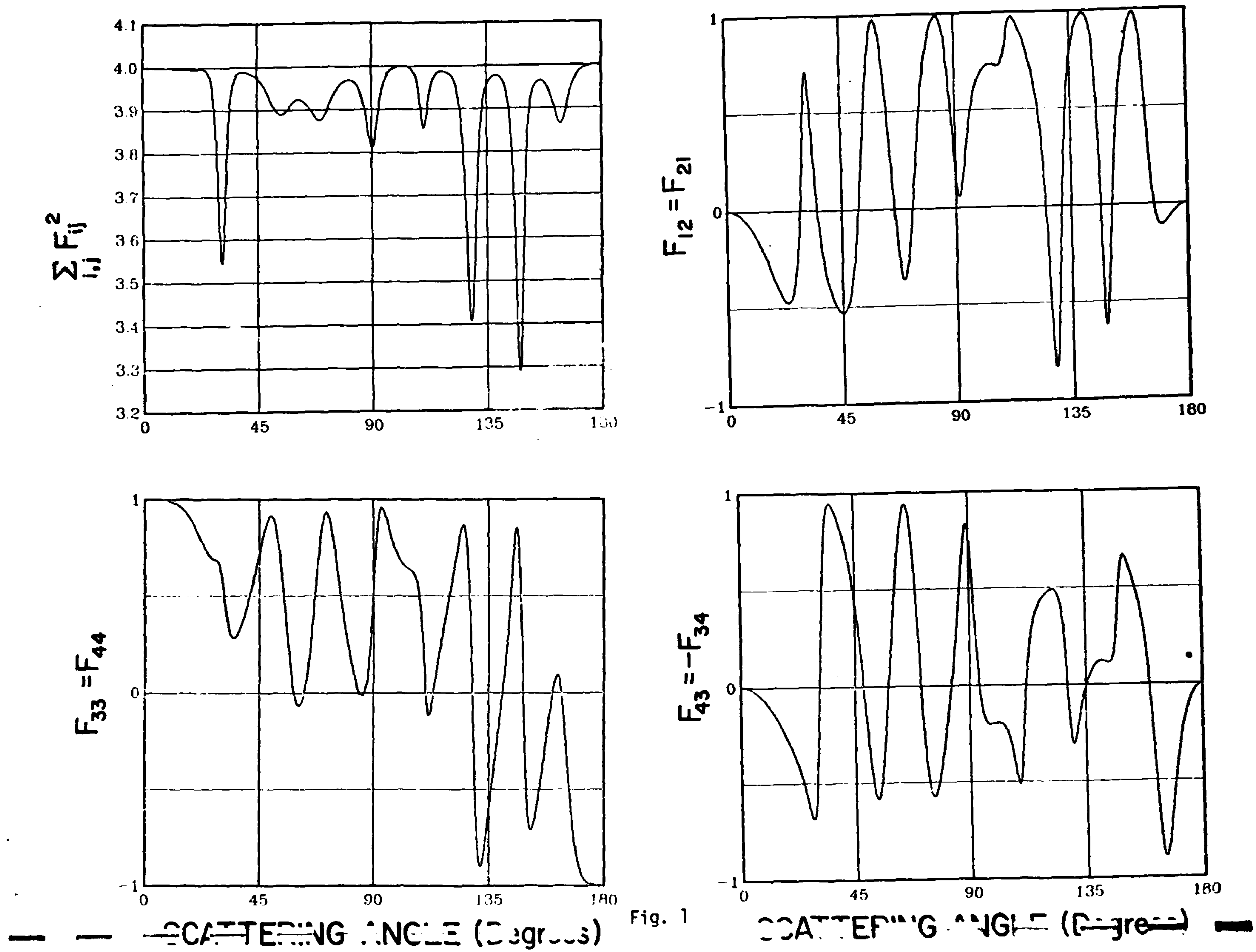
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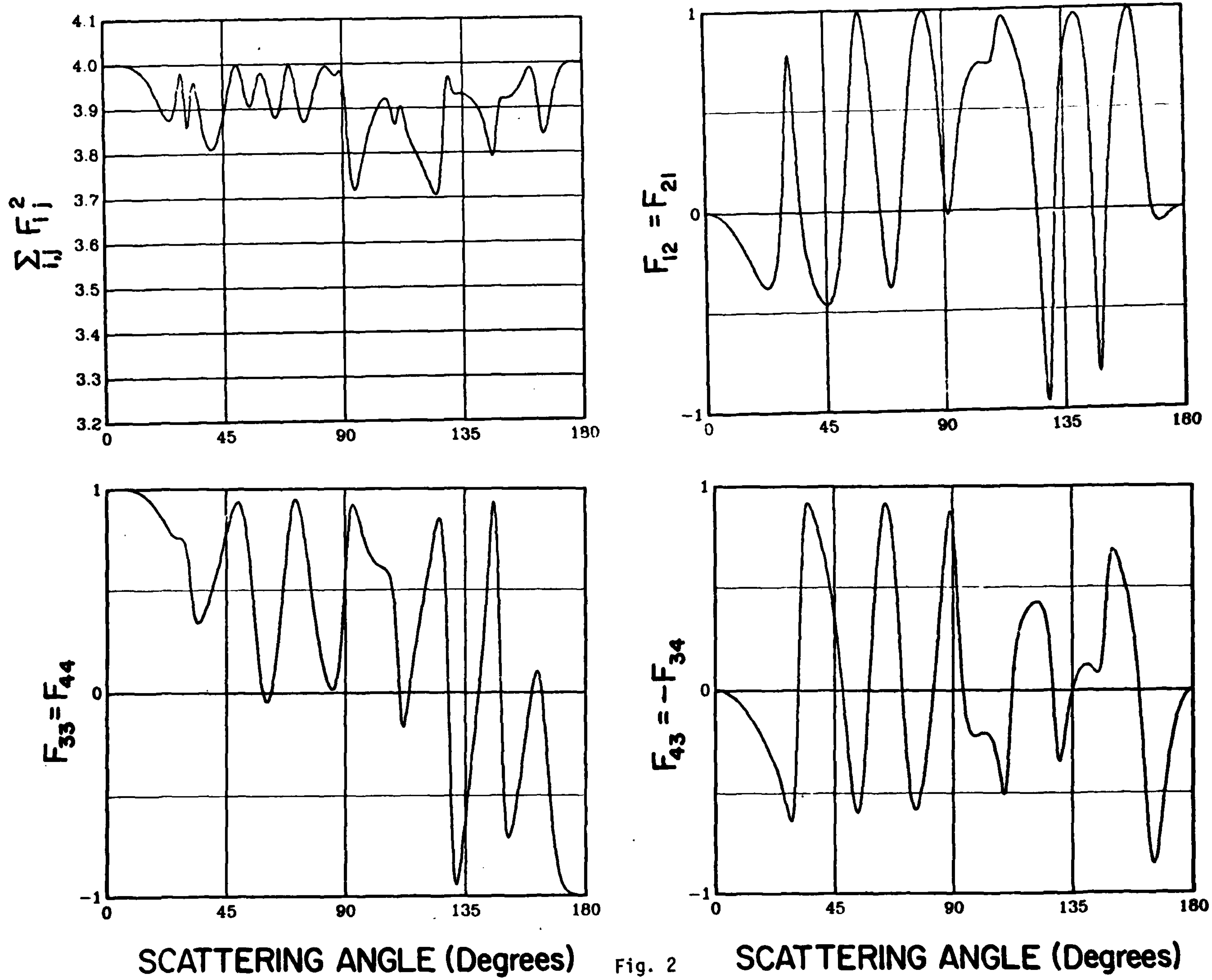
Fig. 1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SCATTERING ANGLE (Degrees)
SCATTERING ANGLE (Degrees)

Fig. 2